On the Determination of Proper Time

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Abstract

Through the analysis of the definition of the duration of proper time of a particle given by the length of its world line, we show that there is no transitivity of the coordinate time function derived from the definition, so there exists an ambiguity in the determination of the duration of the proper time for the particle. Its physical consequence is illustrated with quantum measurement effect.

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The proper time of a free particle is referred to the reading of the clock attached on it. For a particle moving on spacetime manifold $(M, g_{\mu\nu})$, where the world line elment ds is given as[1]

$$ds^2 = g_{00}(x)(dx^0)^2 - g_{ij}(x)(dx^i dx^j), \tag{1}$$

for $i, j = 1, 2, 3$, the duration of its proper time is defined by the length of the world line element

$$\tau = \int_{\tau 0}^{\tau} ds = \int_{t0}^{t} (h_{00}(x^0) - h_{ij}(x^0)\frac{dx^i}{dx^0}\frac{dx^j}{dx^0})^{\frac{1}{2}} dx^0, \tag{2}$$

where $h_{00}(x^0) = g_{00}(x^0, x^i(x^0)), h_{ij}(x^0) = g_{ij}(x^0, x^i(x^0))$, and $x^i = x^i(x^0)$ is the trajectory of the particle observed in the coordinate system $\Sigma : (x^0, x^1, x^2, x^3)$. Eq.(2) indicates that the measure of the duration of proper time should be independent of the choice of coordinate system and, therefore, it also leads to the relation of the coordinate times between any a pair of coordinate system Σ and Σ':

$$\int_{t_0}^{t} (h_{00}(x_0) - h_{ij}(x^0)\frac{dx^i}{dx^0}\frac{dx^j}{dx^0})^{\frac{1}{2}} dx^0$$

$$= \int_{t_0'}^{t'} (h_{0'0'}\left((x^0)'\right) - h_{i'j'}\left((x^0)'\right)\frac{(dx^i)'}{(dx^0)'}\frac{(dx^j)'}{(dx^0)'})^{\frac{1}{2}} (dx^0)', \tag{3}$$

which defines a coordinate time function $t = t(t')$.

For consistency the above-mentioned coordinate time function should be transitive, i.e. if there are unique $t'(t)$, $t''(t')$ and $t''(t)$ between any pair of three systems, we will have $t''\left(t'(t)\right) = t''(t)$. However, we will prove in this letter that the relation of proper times given by Eq.(3) is not transitive, so the matter is much more complicate than we previously assumed, when we define the the duration of proper time, an invariant quantity for all observors.

First we prove the existence of coordinate time function. Eq.(3) is an implicit function of the form: $F(t, t') = 0$. The range (t_0, t) is divided into the union of infinitely

[1] In this paper we always assume that the temporal and spatial parts of world line element are separable.

many small ones $[t_0,t_1)\cup(t_1,t_2)\cdots\cup(t_{n-1},t]$. Implicit function theorem gives a unique function $t'=f_i(t)$ in a small neighborhood (t_i,t_{i+2}), as long as

$$\frac{\partial F(t,t')}{\partial t'}=(h_{0'0'}\left((x^0)'\right)-h_{i'j'}\left((x^0)'\right)\frac{(dx^i)'}{(dx^0)'}\frac{(dx^j)'}{(dx^0)'})^{\frac{1}{2}}\neq 0. \tag{4}$$

By the generalized gluing lemma proved in Appendix we immediately obtain a unique continuous time function $t'=t'(t)$ in the range (t_0,t) and, moreover, it satisfies

$$\frac{dt'}{dt}=(h_{00}(t)-h_{ij}(t)\frac{dx^i}{dt}\frac{dx^j}{dt})^{\frac{1}{2}}/(h_{0'0'}(t')-h_{i'j'}(t')\frac{(dx^i)'}{dt'}\frac{(dx^j)'}{dt'})^{\frac{1}{2}} \tag{5}$$

at every point of the range $[t_0,t]\times[t'_0,t']$.

Suppose there are three different systems Σ^1,Σ^2, and Σ^3 used to observe the motion of a particle. Without the loss of generality σ^1 is the system set up by the co-moving observor of the particle. From the previous results we can obtain the coordinate time relation between Σ^1,Σ^2 as

$$\int_{t_0}^{t}(h_{00}^{(1)}(x_1^0))^{\frac{1}{2}}dx_1^0=\int_{t'_0}^{t'}(h_{00}^{(2)}(x_2^0)-h_{ij}^{(2)}(x_2^0)\frac{dx_2^i}{dx_2^0}\frac{dx_2^j}{dx_2^0})^{\frac{1}{2}}dx_{(2)}^0, \tag{6}$$

and that between Σ^1,Σ^3 as

$$\int_{t_0}^{t}(h_{00}^{(1)}(x_1^0))^{\frac{1}{2}}dx_1^0=\int_{t''_0}^{t''}(h_{00}^{(3)}(x_3^0)-h_{ij}^{(3)}(x_3^0)\frac{dx_3^i}{dx_3^0}\frac{dx_3^j}{dx_3^0})^{\frac{1}{2}}dx_3^0. \tag{7}$$

On the other hand, there is the relation of the world line element between the co-moving system of Σ^2 and the system Σ^3:

$$g_{00}^{(2)}(x_2)(dx_2^0)^2=g_{00}^{(3)}(x_3)(dx_3^0)^2-g_{ij}^{(3)}(x_3)dx_3^i dx_3^j. \tag{8}$$

Substituting Eq.(8) into Eq.(6), we obtain the following coordinate time relation:

$$\int_{t_0}^{t}(h_{00}^{(1)}(x_1^0))^{\frac{1}{2}}dx_1^0=\int_{s'_0}^{s'}(g_{00}^{(3)}(x_3)(dx_3^0)^2-g_{ij}^{(3)}(x_3)dx_3^i dx_3^j-g_{ij}^{(2)}(x_2)dx_2^i dx_2^j)^{\frac{1}{2}}$$

$$=\int_{t''_0}^{t''}(h_{00}^{(3)}(x_3^0)-h_{ij}^{(3)}(x_3^0)\frac{dx_3^i}{dx_3^0}\frac{dx_3^j}{dx_3^0}-h_{ij}^{(2)}(x_2^0(x_3^0))\frac{dx_2^i}{dx_2^0}\frac{dx_2^j}{dx_2^0}\frac{dx_2^0}{dx_3^0}\frac{dx_2^0}{dx_3^0})^{\frac{1}{2}}dx_3^0. \tag{9}$$

With the relation in Eq.(5) between Σ^2 and Σ^3, we finally get another implicit coordinate time function:

$$\int_{t_0}^{t}(h_{00}^{(1)}(x_1^0))^{\frac{1}{2}}dx_1^0=\int_{t''_0}^{t''}(A(x_3^0)(h_{00}^{(3)}(x_3^0)-h_{ij}^{(3)}(x_3^0)\frac{dx_3^i}{dx_3^0}\frac{dx_3^j}{dx_3^0}))^{\frac{1}{2}}\ dx_3^0, \tag{10}$$

where $A(x_3^0) = (h_{00}^{(2)}(x_2^0(x_3^0)))/(h_{00}^{(2)}(x_2^0(x_3^0)) - h_{ij}^{(2)}(x_2^0(x_3^0))\frac{dx_2^i}{dx_2^0}(x_2^0(x_3^0))\frac{dx_2^j}{dx_2^0}(x_2^0(x_3^0))) \neq 1$, between Σ^1 and Σ^3. $x_2^0(x_3^0)$ here is the coordinate time function between Σ^2 and Σ^3. Obviously, with the factor $A(x_3^0)$, it is a different coordinate time function from Eq.(7), and it goes against the uniqueness of the coordinate time function between a pair of coordinate system. Therefore, we conclude that is no associativity for the coordinate time function; it is a logical flaw in the definition of proper time by the world line length.

A point implied in the above discussion is he correspondence of integral domain of world line length in different systems; it requires that range $[t_0, t]$, $[t'_0, t']$ and $[t''_0, t'']$ are exactly in one-to-one correspondence in the calculation of the duration of proper time for the observed particle in these systems. It is the only postulates we work on, and we call it **Principle of Measurement Correspondence**.

In terms of the proper time of the observors this principle can be formulated as follows: The duration of a particular physical process occupies a definite range of the proper time axis of an observor, except for the choice of proper time origin and the ratio of ticking rate of clocks which is constant according to the 'hypothesis of consistency'[1]. Therefore, for any a couple of different observors, there is a unique function $\tau' = f(\tau)$ to relate the measures of the proper times, $[\tau_1, \tau_2]$ and $[\tau'_1, \tau'_2]$, they spend for the observation of the process.

To illustrate the physical implication of the principle, we set up an imaginary experiment which involves quantum measurement correlation (EPR effect). Suppose a electron-positron couple with total spin zero is created but fails to form a bound state in an inertial system Σ in Minkovski spacetime. Their classical trajectories are therefore as follows:

$$x^e = vt,$$

$$x^p = -vt, \tag{11}$$

if Σ happen to be the mass center system. Let Σ^1, Σ^2 the co-moving system of the electron and positron and the clocks attached to them are set zero simultaneously

with the clock in Σ at the moment the electron-positron couple is created. The wave functions of the particles in the co-moving systems are the follows[2]:

$$\psi_s^{e,(1)}(x') = \sqrt{2m} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} e^{-imt} \quad (s = \pm\frac{1}{2}), \tag{12}$$

and

$$\psi_s^{p,(2)}(x'') = -\sqrt{2m} \begin{pmatrix} 0 \\ \varepsilon\chi_s \end{pmatrix} e^{imt} \quad (s = \pm\frac{1}{2}), \tag{13}$$

where $\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In system Σ, their wave functions, which satisfy Dirac equation in a general system

$$(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m)\psi(x) = 0, \tag{14}$$

can be obtained through Lorentz transformation:

$$\psi_s^e(x) = S^{-1}(\Lambda(\beta))\psi_s^{e,(1)}(x'), \tag{15}$$

$$\psi_s^p(x) = S^{-1}(\Lambda(-\beta))\psi_s^{p,(2)}(x''), \tag{16}$$

where $\Lambda_\nu^\mu = \delta_\nu^\mu + h_\nu^\mu$ is Lorentz transformation matrix of the coordinates and

$$S(\Lambda) = exp(-\frac{i}{4}h^{\mu\nu}\sigma_{\mu\nu}) = exp(\frac{1}{8}h^{\mu\nu}[\gamma_\mu, \gamma_\nu]).$$

The total wave function of the spin-zero system in Σ before any measurement to determine the spin state of any of the particles is

$$\Psi(x) = \frac{1}{\sqrt{2}}(\psi^e_{+\frac{1}{2}}\psi^p_{-\frac{1}{2}} - \psi^e_{-\frac{1}{2}}\psi^p_{+\frac{1}{2}})(x). \tag{17}$$

At a particular moment T_0 in Σ, which corresponds to

$$T_0' = \gamma(T_0 - \frac{v^2}{c^2}T_0) = \sqrt{1 - \frac{v^2}{c^2}}\, T_0,$$

in Σ^1, the observor in Σ^1 undertake a measurement of the spin state of the electron with the result, say $s = +\frac{1}{2}$, then the total wave function will collapse to $\psi^{e,(1)}_{+\frac{1}{2}}(x')\psi^{p,(1)}_{-\frac{1}{2}}(x')$

immediately, because of quantum measurement correlation (EPR effect). Therefore, in Σ^1, the total wave function since the creation the electron-positron couple can be given as

$$\Psi^{(1)}(x') = \frac{1}{\sqrt{2}}(\theta(t') - \theta(t' - T'_0))(\psi^e_{+\frac{1}{2}}\psi^p_{-\frac{1}{2}} - \psi^e_{-\frac{1}{2}}\psi^p_{+\frac{1}{2}})(x')$$

$$+ \theta(t' - T'_0)\psi^e_{+\frac{1}{2}}\psi^p_{-\frac{1}{2}}(x'), \tag{18}$$

where $\theta(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$ In Σ, according to Lorentz transformation Eq.(15), the correspondent wave function becomes

$$\Psi_1(x) = (\theta(\sqrt{1 - \frac{v^2}{c^2}}\, t) - \theta(\sqrt{1 - \frac{v^2}{c^2}}\, t - \sqrt{1 - \frac{v^2}{c^2}}\, T_0))\Psi(x)$$

$$+ \theta(\sqrt{1 - \frac{v^2}{c^2}}\, t - \sqrt{1 - \frac{v^2}{c^2}}\, T_0)S^{-1}(\Lambda(\beta))\psi^{e,(1)}_{+\frac{1}{2}}\psi^{p,(1)}_{-\frac{1}{2}}(x'). \tag{19}$$

Obviously, the moment of wave function collapse is determined by

$$\sqrt{1 - \frac{v^2}{c^2}}\, t_1 - \sqrt{1 - \frac{v^2}{c^2}}\, T_0 = 0,$$

i.e. $t_1 = T_0$.

On the other hand, the coordinate time function between Σ^1 and Σ^2 is given by

$$t'' = \frac{t' - (-\frac{u}{c^2})(-ut')}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{1 - \frac{v^2}{c^2}}{1 + \frac{v^2}{c^2}}\, t', \tag{20}$$

since the relative velocity of Σ^2 to Σ^1 is $u = \frac{-2v}{1+\frac{v^2}{c^2}}$. The total wave function in Σ^2 can be obtained by the Lorentz transformation from Σ^1 to Σ^2 as

$$\Psi^{(2)}(x'') = ((\theta(t'') - \theta(t'' - \frac{1 - \frac{v^2}{c^2}}{1 + \frac{v^2}{c^2}}\, T'_0))S(\Lambda(-\frac{u}{c}))S(\Lambda(\frac{v}{c}))\Psi(x)$$

$$+ \theta(t'' - \frac{1 - \frac{v^2}{c^2}}{1 + \frac{v^2}{c^2}}\, T'_0))S(\Lambda(-\frac{u}{c}))\psi^{e,(1)}_{+\frac{1}{2}}(x')\psi^{p,(1)}_{-\frac{1}{2}}(x'). \tag{21}$$

With the consideration of the coordinate time function between Σ and Σ^2

$$t'' = \gamma(t - (-\frac{v}{c})(-\frac{v}{c})t), \tag{22}$$

the wave function $\Psi^{(2)}(x'')$ is transformed back to

$$\Psi_2(x) = (\theta(\sqrt{1-\frac{v^2}{c^2}}\ t) - \theta(\sqrt{1-\frac{v^2}{c^2}}\ t - \frac{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}{1+\frac{v^2}{c^2}}\ T_0))\Psi(x)$$

$$+\,\theta(\sqrt{1-\frac{v^2}{c^2}}\ t - \frac{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}{1+\frac{v^2}{c^2}}\ T_0))S^{-1}(\Lambda(-\frac{v}{c}))S(\Lambda(-\frac{u}{c}))\psi^{e,(1)}_{+\frac{1}{2}}(x')\psi^{p,(1)}_{-\frac{1}{2}}(x'), \qquad (23)$$

which gives us another time for the collapse of wave function at $t_2 = \frac{1-\frac{v^2}{c^2}}{1+\frac{v^2}{c^2}}\ T_0$ in Σ.

An absolute event (the operation of spin state measurement) in Σ^1 gives rise to two different correspondences in Σ, and leads to the ambiguity of the determination of proper time in Σ^2. In fact, it reflects the contradiction of the relativity of simultaneity with quantum measurement correlation effect. This phenomenon deserves further study on the problem, if we desire to find a consistent theory.

Appendix: Proof of the generalized glupico time.tex ing lemma

Gluing lemma[3] states that if there is $f : A \to B$ with

$$f(x) = \begin{cases} f_1(x) & x \in A_1, \\ f_2(x) & x \in A_2, \end{cases}$$

where $A = A_1 \cup A_2$, the union of two open sets, and f_1 and f_2 are continuous respectively, then $f(x)$ is continuous in the whole open set A, as long as $f_1(x) = f_2(x)$ for $x \in A_1 \cap A_2$.

We need to generalize the statement to the case where the range $[t_0, t]$ is covered by a family of countably infinite open set plus two semi-open sets at the end points: $[t_0, t] = [t_0, t_1)\cup(t_1, t_2)\cdots\cup(t_{n-1}, t_n)\cup(t_n, t]$, and there is the implicit function $F(t, t') = 0$ over $[t_0, t] \times [t' - 0, t']$. The open set (t_i, t_{i+2}) is so small that implicit function theorm guarantees a unique differentiable function $f_i(x)$ on it. From the uniqueness of these functions we have $f_i(x) = f_{i+1}(x)$ for $x \in (t_{i+1}, t_{i+2})$. Let C be an open set in $[t'_0, t']$ and t the function which equals f_i on every these open set, we have

$$t^{-1}(C) \cap [t_0, t] = f_0^{-1}(C \cap [t_0, t_2)) \cup f_1^{-1}(C \cap (t_1, t_3)) \cdots \cup f_{n-1}^{-1}(C \cap (t_{n-1}, t]).$$

It is an open set because the countably infinite union of open sets is an open set, hence the continuity of function t.

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